System of viscous conservation laws:

$$u_t + f(u)_x = \mu u_{xx}$$

Profile:

$$u^*(x,t) = \phi(x \Leftrightarrow st)$$
$$\mu\phi' = h(\phi) = f(\phi) \Leftrightarrow s\phi \Leftrightarrow q, \quad \phi(\pm\infty) = u_\pm$$

Integrated perturbation:

$$U(x \Leftrightarrow st, t) = \int_{-\infty}^{x} u(\xi, t) \Leftrightarrow \phi(\xi \Leftrightarrow st)\, d\xi$$

Integrated equation:

$$U_t + h'(\phi)U_x \Leftrightarrow \mu U_{xx} = F(\phi, U_x)$$

where $h' = f' \Leftrightarrow sI$

Main task: Proof

$$||U(\cdot,T)||_{L^2}^2 + \int_0^T ||U_x||_{L^2}^2\, dt \leq C||U(\cdot,0)||_{L^2}^2$$

($\rightsquigarrow$ stability).

Stability of profiles of general small-amplitude Laxian shock waves

(via Weighted Energy Estimates)

1. Viscous Shock Profiles

Hyperbolic conservation law

$$u_t + f(u)_x = \mu u_{xx} \quad x \in \mathbb{R},\ u \in \mathbb{R}^n \quad (1)$$
$$u(\pm\infty, t) = u_\pm$$

(i.e. $f'(u)$ $\mathbb{R}$-diagonizable $\forall u$).

Consider a traveling wave solution

$$u^*(x,t) = \phi(x \Leftrightarrow st)$$

i.e., ϕ solves

$$\mu\phi' = h(\phi) = f(\phi) \Leftrightarrow s\phi \Leftrightarrow q \quad , \quad \phi(\pm\infty) = u_\pm \ ,$$

where u_-, u_+, s, q satisfy

$$f(u_-) \Leftrightarrow su_- = f(u_+) \Leftrightarrow su_+ = q \quad ,$$

Definition

$u^*(x,t) = \phi(x \Leftrightarrow st)$ asymptotically stable

:$\Leftrightarrow$ $\exists$ $(B, ||\cdot||_B)$, $\beta > 0$: $\forall$ $u_0 \in B$, $||u_0||_B < \beta$:

Solution u of (1) with perturbed data

(S) $u(\cdot,0) = \phi + u_0$ exists for all $t > 0$ and has

$\lim_{t\to\infty} \sup_x |u(x,t) \Leftrightarrow \phi(x \Leftrightarrow st)| = 0$

Theorem (Goodman, 1985)

- λ simple eigenvalue of f'
- "convex": $\mathbb{R} \cdot r = \ker(f' \Leftrightarrow \lambda I) \Rightarrow r \cdot \nabla\lambda \neq 0$
- ϕ associated with λ: $\lambda(u_-) > s > \lambda(u_+)$
- small amplitude: $|u_+ \Leftrightarrow u_-| << 1$

$\Rightarrow$ (S) for all $u_0 \in L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} u_0 \, dx = 0$

and $||U_0||_{H^2} < \beta$ ($U_0(x) := \int_{-\infty}^{x} u_0(y)dy$)

Goal:

Theorem 1

Same without "convex"

2. The Integrated Equation

Subtract the solution $u^*(x,t) = \phi(x \Leftrightarrow st)$ from a solution u (with perturbed initial data $\phi + (U_0)_x$):

$$(u \Leftrightarrow u^*)_t + (f(u) \Leftrightarrow f(u^*))_x = \mu(u \Leftrightarrow u^*)_{xx}$$

Integrated perturbation:

$$U(x \Leftrightarrow st, t) = \int_{-\infty}^{x} u(x,t) \Leftrightarrow \phi(x \Leftrightarrow st)\, dx$$

Integrate and change to moving coordinates $(x \Leftrightarrow st, t)$):

$$\Leftrightarrow sU_x + U_t + f(\phi + U_x) \Leftrightarrow f(\phi) = \mu U_{xx}$$
$$U(x,0) = U_0(x) = \int_{-\infty}^{x} u(x,0) \Leftrightarrow \phi(x)\, dx$$

Using Taylor expansion

$$f(\phi + U_x) \Leftrightarrow f(\phi) = f'(\phi)U_x \Leftrightarrow F(\phi, U_x)$$

Integrated Equation

$$U_t + h'(\phi)U_x \Leftrightarrow \mu U_{xx} = F(\phi, U_x) \qquad (2)$$
$$U(\cdot, 0) = U_0 = \int_{-\infty}^{x} u(x,0) \Leftrightarrow \phi(x)\, dx$$

where $h = f \Leftrightarrow s\mathrm{id}$.

3. Example: Scalar case, convex flux

Apply a standard energy estimate to the integrated equation

$$U_t + h'(\phi)U_x \Leftrightarrow \mu U_{xx} = F(\phi, U_x)$$

Multiply by U and integrate $\int_{-\infty}^{\infty} dx$, $\int_0^T dt$:

$$||U(\cdot, T)||_{L^2}^2 + \int_0^T \int_{-\infty}^{\infty} \Leftrightarrow (h'(\phi))_x U^2 \, dx \, dt$$
$$+ \int_0^T ||U_x||_{L^2}^2 \, dt \leq C||U(\cdot, 0)||_{L^2}^2$$

proceeding in a similar (but less restrictive) way with U_x and U_{xx} we have

Proposition 1

If $n = 1$ (scalar equation) and $(h'(\phi))_x < 0$ then

$$||U(\cdot, T)||_{H^{2,2}}^2 + \int_0^T ||U_x||_{H^{2,2}}^2 \, dt \leq C||U(\cdot, 0)||_{H^{2,2}}^2$$

This a-priori estimate ensures the existence of U for all times and gives a decay for U_x such that Theorem 1 holds.

4. Scalar case, non-convex flux (Matsumura & Nishihara 1994):

Multiply by $U \cdot w$, $w = w(x)$ and integrate $\int_{-\infty}^{\infty} dx$:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}(w|U|^2)\, dx + \int_{-\infty}^{\infty} U w h'(\phi) U_x + \mu U w_x U_x$$
$$+\mu w (U_x)^2 \Leftrightarrow U w F(\phi, U_x)\, dx = 0$$

Find positive weight w such that:

$$\Leftrightarrow \frac{1}{2}\Big(w h'(\phi) + \mu w_x\Big)_x > 0$$

Ansatz $w(x) = \tilde{w}(\phi(x))$:

$$\begin{aligned} & \Leftrightarrow \frac{1}{2}\Big(w h'(\phi) + \mu w_x\Big)_x \\ = & \Leftrightarrow \frac{1}{2}\Big(\tilde{w}(\phi) h'(\phi) + \mu \tilde{w}'(\phi)\phi_x\Big)_x \\ = & \Leftrightarrow \frac{1}{2}\Big((\tilde{w}h)'(\phi)\Big)_x = \Leftrightarrow \frac{1}{2}(\tilde{w}h)''(\phi)\phi_x \end{aligned}$$

Choose

$$\tilde{w}(u) = \Leftrightarrow \frac{(u \Leftrightarrow u_+)(u \Leftrightarrow u_-)}{h} \cdot \text{sign}\phi_x > 0$$

to obtain

$$\Leftrightarrow \frac{1}{2}\Big(w h'(\phi) + \mu w_x\Big)_x = |\phi_x|$$

5. System case, non-convex mode

Diagonalize (Goodman 1985): Matrix function

$$L(x) = \tilde{L}(\phi(x)),\ R(x) = \tilde{R}(\phi(x))$$

such that $LR \equiv I$ and:

$$L(h'(\phi))R = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

where

$$\lambda_p = \lambda \Leftrightarrow s\,, \quad \lambda_i < 0 < \lambda_j\ (i < p < j)$$

Substitute $U =: RV$ in (2), multiply by $V^T WL$, integrate $\int_{-\infty}^{\infty} dx \ \rightsquigarrow$

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}(V^T WV) + V^T W\Lambda V_x + V^T W\Lambda LR_x V$$
$$+\mu(V^T WL)_x(RV)_x \Leftrightarrow V^T WLF(\phi, (RV)_x)\ dx$$
$$= 0$$

Choose

$$W = \text{diag}(1, \ldots, 1, w, 1, \ldots, 1)$$

Group terms:

(A1) $\frac{1}{2}\frac{\partial}{\partial t}(V^T W V)$

(A2) $(w\lambda_p + \mu w_x)V_p(V_p)_x$

(A3) $\sum_{k\neq p}(\Leftrightarrow\frac{1}{2}(\lambda_k)_x + l_k(r_k)_x\lambda_k)(V_k)^2$

(A4) $\mu w((V_p)_x)^2 + \mu\sum_{k\neq p}((V_k)_x)^2$

(B1) $\sum_{j\neq p}(w\lambda_p + \mu w_x)l_p(r_j)_x V_p V_j$

(B2) $\sum_{i\neq p, i\neq j}\lambda_i l_i(r_j)_x V_i V_j$

(B3) $\mu V^T W L_x R V_x + \mu V_x^T W L R_x V$

(B4) $\mu V^T W L_x R_x V$

(B5) $\Leftrightarrow V^T W L F(\phi, (RV)_x)$

Remarks (on the diagonalization):

1. $|L_x|$, $|R_x| \le O(1)|\phi_x| \le O(1)|u_+ \Leftrightarrow u_-|$.

2. If l_k, r_k are left and right eigenvectors of h' ($l_i r_j = \delta_{ij}$), so are $\frac{1}{\alpha_k} l_k$ and $\alpha_k r_k$, i.e. there are n degrees of freedom in our choice of L and R.

For $k \neq p$, use α_k to achieve positivity in the V_k-component:

$$(\text{A3}) = (\underbrace{\Leftrightarrow\frac{1}{2}(\lambda_k)_x}_{|\cdot| \le C|\phi_x|} + \underbrace{\frac{(\alpha_k)_x}{\alpha_k}\lambda_k}_{\ge 2C|\phi_x|})(V_k)^2$$

by choice of α_k
(as λ_k uniformly
away from 0)

Not so for $k = p$ (as λ_p crosses 0 along ϕ).

For $k = p$, use w.

Lemma

$\epsilon := |u_- \Leftrightarrow u_+| << 1$ then
$\exists\, w : \mathbb{R} \to \mathbb{R}$ with $\inf_x w(x)$, $\inf_x (1/w(x)) > 0$ ($\rightsquigarrow$ L_2-norm in (A1), (A4) and estimate for (B5)) and

$$\Leftrightarrow \frac{1}{2}(w\lambda_p + \mu w_x)_x \;=\; |\phi_x|$$

($\rightsquigarrow$ positivity of (A2))

$$|w\lambda_p + \mu w_x| \;\leq\; 4|u_+ \Leftrightarrow u_-|$$

($\rightsquigarrow$ estimate for (B1))

$$|\mu(w\phi_x)_x| = |(wh(\phi))_x| \;\leq\; 4|u_+ \Leftrightarrow u_-| \cdot |\phi_x|$$

($\rightsquigarrow$ estimate for (B3))

$$|\mu w\phi_x| = |wh(\phi)| \;\leq\; 8|u_+ \Leftrightarrow u_-|^2$$

($\rightsquigarrow$ estimate for (B4))

Proof of Lemma

Analyze the differential equation

$$\mu w_x + w\lambda_p = \int_{x_0}^{x} |\phi_x| d\xi \quad , \quad w(0) = w_0$$

and choose the parameters x_0, w_0 appropriately.

Proof of Theorem

Apply Lemma and estimate (B1) to (B5)